生产实习答辩

稀薄气体动力学的数值模拟

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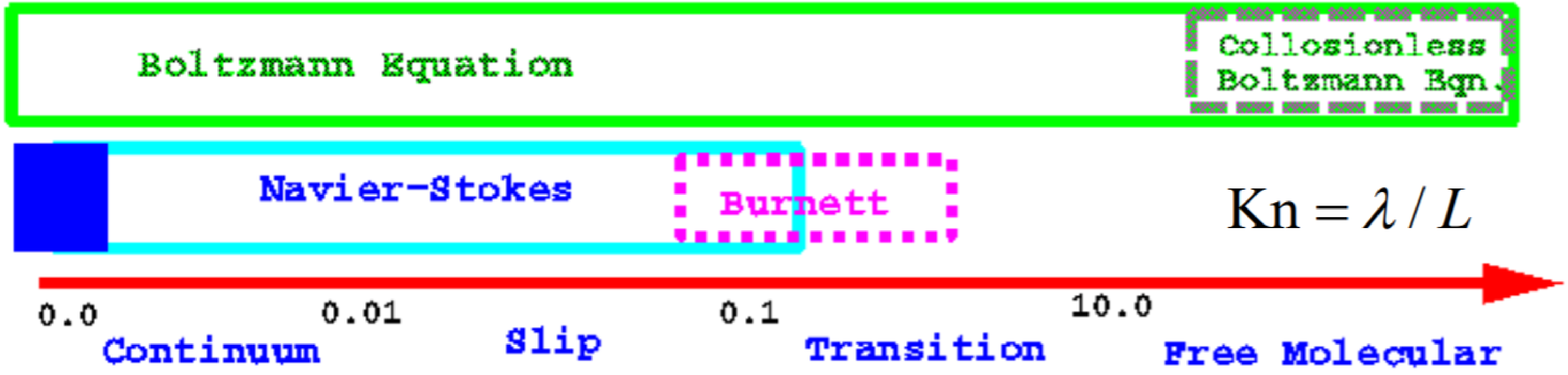
1. 背景介绍——稀薄气体

- 气体密度很低时，流体力学中的连续介质假设不再适用，气体分子离散特性开始显现
- 衡量气体稀薄程度的无量纲参数：克努森数（Kn）
- $Kn = \lambda/L$：气体分子的平均自由程与流场的几何尺度之比



1. 背景介绍——玻尔兹曼方程

- Kn比较大时，N-S方程不成立，使用玻尔兹曼方程描述分子的运动
- 分子运动的分布函数：$f(x_i, t, c_i)$
- 玻尔兹曼方程为：

$$\frac{\partial f}{\partial t} + c_k \frac{\partial f}{\partial x_k} = S(f)$$

- $S(f)$：碰撞项，反映了分子相互碰撞引起的分布函数的变化

1. 背景介绍——分布函数的矩

- 对分布函数取矩，即对分布函数在速度空间上积分，得到新的变量：

$$\rho = m\int f d\mathbf{c}$$

$$\rho v_i = m\int c_i f d\mathbf{c}$$

$$\rho\varepsilon = \frac{3}{2}\rho\frac{k}{m}T = \frac{3}{2}\rho\theta = \frac{m}{2}\int c^2 f d\mathbf{c}$$

$$\sigma_{ij} = m\int c_{\langle i}c_{j\rangle} f d\mathbf{c}$$

$$q_i = \frac{m}{2}\int c^2 c_i f d\mathbf{c}$$

- ρ-密度，v-速度，θ-温度， σ –粘性应力张量， q-热流密度

1. 背景介绍——R13方程

根据各阶矩的形式，对玻尔兹曼方程进行积分

- 连续方程、动量方程以及能量方程为：

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}+\rho\frac{\partial v_k}{\partial x_k}=0,$$

$$\rho\frac{\mathrm{d}v_i}{\mathrm{d}t}+\theta\frac{\partial\rho}{\partial x_i}+\rho\frac{\partial\theta}{\partial x_i}+\frac{\partial\sigma_{ik}}{\partial x_k}=0,$$

$$\frac{3}{2}\rho\frac{\mathrm{d}\theta}{\mathrm{d}t}+\rho\theta\frac{\partial v_k}{\partial x_k}+\frac{\partial q_k}{\partial x_k}+\sigma_{kl}\frac{\partial v_k}{\partial x_l}=0,$$

- 方程中有σ和q，为此需要引入更高阶的积分方程

1. 背景介绍——R13方程

- 粘性应力张量及热流密度满足的方程为:

$$\frac{\mathrm{d}\sigma_{ij}}{\mathrm{d}t}+\sigma_{ij}\frac{\partial v_k}{\partial x_k}+\frac{4}{5}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}}+2\rho\theta\frac{\partial v_{\langle i}}{\partial x_{j\rangle}}+2\sigma_{k\langle i}\frac{\partial v_{j\rangle}}{\partial x_k}+\frac{\partial m_{ijk}^{(\eta)}}{\partial x_k}=\Sigma_{ij}^{(\eta,1)}+\Sigma_{ij}^{(\eta,2)},$$

$$\begin{aligned}&\frac{\mathrm{d}q_i}{\mathrm{d}t}+\frac{5}{2}\rho\theta\frac{\partial\theta}{\partial x_i}+\frac{5}{2}\sigma_{ik}\frac{\partial\theta}{\partial x_k}+\theta\frac{\partial\sigma_{ik}}{\partial x_k}-\theta\sigma_{ik}\frac{\partial\ln\rho}{\partial x_k}+\frac{7}{5}q_k\frac{\partial v_i}{\partial x_k}\\&\quad+\frac{2}{5}q_k\frac{\partial v_k}{\partial x_i}+\frac{7}{5}q_i\frac{\partial v_k}{\partial x_k}-\frac{\sigma_{ij}}{\rho}\frac{\partial\sigma_{jk}}{\partial x_k}\\&\quad+C^{(\eta)}\left(\sigma_{ik}\frac{\partial\theta}{\partial x_k}+\theta\frac{\partial\sigma_{ik}}{\partial x_k}\right)+\frac{1}{2}\frac{\partial R_{ik}^{(\eta)}}{\partial x_k}+\frac{1}{6}\frac{\partial\Delta^{(\eta)}}{\partial x_i}+m_{ijk}^{(\eta)}\frac{\partial v_j}{\partial x_k}=Q_i^{(\eta,1)}+Q_i^{(\eta,2)},\end{aligned}$$

- 以上方程组共有13个独立变量（ρ + v 3个分量+ θ + σ 5个独立分量+ q 3个分量），因此称为R13方程（Regularized 13-moment equations）

1. 背景介绍——R13方程的封闭性

- 方程含有高阶矩m_{ijk}，R_{ij}，Δ，以及碰撞项$\Sigma_{ij}^{(\eta,1)}$，$\Sigma_{ij}^{(\eta,2)}$和$Q_i^{(\eta,1)}$，$Q_i^{(\eta,2)}$则方程组不封闭
- 为使方程组封闭，需要引入高阶矩和碰撞项以低阶矩表示的表达式

1. 背景介绍——R13方程的封闭性

$$m_{ijk}^{(\eta)} = \frac{\mu}{\theta\rho}\left(A_1^{(\eta)} q_{\langle i}\frac{\partial v_j}{\partial x_{k\rangle}} + A_2^{(\eta)}\theta\sigma_{\langle ij}\frac{\partial \ln\rho}{\partial x_{k\rangle}} + A_3^{(\eta)}\sigma_{\langle ij}\frac{\partial\theta}{\partial x_{k\rangle}} + A_4^{(\eta)}\theta\frac{\partial\sigma_{\langle ij}}{\partial x_{k\rangle}}\right),$$

$$\Delta^{(\eta)} = \frac{\mu}{\theta\rho}\left(B_1^{(\eta)}\theta\frac{\partial q_k}{\partial x_k} + B_2^{(\eta)}\theta\sigma_{ij}\frac{\partial v_i}{\partial x_j} + B_3^{(\eta)} q_k\frac{\partial\theta}{\partial x_k} + B_4^{(\eta)}\theta q_k\frac{\partial\ln\rho}{\partial x_k}\right),$$

$$\begin{aligned}R_{ij}^{(\eta)} = C_0^{(\eta)}\theta\sigma_{ij} + \frac{\mu}{\theta\rho}\Bigg(&C_1^{(\eta)}\theta\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + C_2^{(\eta)}\theta\sigma_{k\langle i}\frac{\partial v_{j\rangle}}{\partial x_k} + C_3^{(\eta)}\theta^2\rho\frac{\partial v_{\langle i}}{\partial x_{j\rangle}}\\ &+ C_4^{(\eta)} q_{\langle i}\frac{\partial\theta}{\partial x_{j\rangle}} + C_5^{(\eta)}\theta q_{\langle i}\frac{\partial\ln\rho}{\partial x_{j\rangle}}\Bigg).\end{aligned}$$

$$\begin{aligned}\Sigma_{ij}^{(\eta,1)} = {}&D_0^{(\eta)}\frac{\theta\rho}{\mu}\sigma_{ij} + D_1^{(\eta)}\left(\frac{\partial v_{\langle i}}{\partial x_k}\sigma_{j\rangle k} + \frac{\partial v_k}{\partial x_{\langle i}}\sigma_{j\rangle k}\right) + D_2^{(\eta)}\frac{\partial v_k}{\partial x_k}\sigma_{ij} + D_3^{(\eta)}\rho\theta\frac{\partial v_{\langle i}}{\partial x_{j\rangle}},\\ &+ D_4^{(\eta)} q_{\langle i}\frac{\partial\ln\theta}{\partial x_{j\rangle}} + D_5^{(\eta)} q_{\langle i}\frac{\partial\ln\rho}{\partial x_{j\rangle}} + D_6^{(\eta)}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}},\end{aligned}$$

$$\begin{aligned}Q_i^{(\eta,1)} = {}&E_0^{(\eta)}\frac{\theta\rho}{\mu}q_i + E_1^{(\eta)}\sigma_{ik}\frac{\partial\theta}{\partial x_k} + E_2^{(\eta)}\theta\sigma_{ik}\frac{\partial\ln\rho}{\partial x_k} + E_3^{(\eta)} q_k\left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k}\right)\\ &+ E_4^{(\eta)} q_i\frac{\partial v_k}{\partial x_k} + E_5^{(\eta)}\theta\frac{\partial\sigma_{ki}}{\partial x_k} + E_6^{(\eta)}\theta\rho\frac{\partial\theta}{\partial x_i},\end{aligned}$$

1. 背景介绍——R13方程的封闭性

- $A_i^{(\eta)}$， $B_i^{(\eta)}$等均为和参数η有关的常数，相关文献中给出了它们的取值
- η为分子模型中的参数， $\eta = 5$为最简单的Maxwell molecules，我们的工作主要关注$\eta = 5$的情况，也涉及了$\eta = 10$的情况
- 将上页的表达式代入R13方程，即完成了方程的封闭

2.工作内容

2. 工作内容——总述

- 文献调研，了解背景知识（见第1部分）
- 二维R13方程的线性化
- 实际问题求解：圆球绕流

2. 工作内容——二维R13方程的线性化

- R13方程仍然很复杂，难以直接求解，需要简化
- 实习单位已经给出R13方程的完整形式（包括$\eta=5$ 和$\eta=10$的情况）
- 文献中已给出一维R13方程的无量纲线性化方法，仿照其进行二维情况的降维和线性化

R13_20_5_q	2020/7/8 21:45	WDX 文件
R13_20_5_rq	2020/7/8 21:50	WDX 文件
R13_20_5_rsigma	2020/7/8 21:50	WDX 文件
R13_20_5_sigma	2020/7/8 21:45	WDX 文件
R13_20_10_q	2020/7/8 21:52	WDX 文件
R13_20_10_rq	2020/7/8 21:52	WDX 文件
R13_20_10_rsigma	2020/7/8 21:52	WDX 文件
R13_20_10_sigma	2020/7/8 21:52	WDX 文件

实习单位给出的包含R13方程完整形式的部分文件

二维R13方程的线性化——线性化准则

- 在线性小扰动假设下，将R13方程中的各个变量在平衡位置附近展开

$$\rho = \rho_0(1+\hat{\rho}), \theta = \theta_0\left(1+\hat{\theta}\right), v_i = \sqrt{\theta_0}\hat{v}_i,$$

$$\sigma_{ij} = \rho_0\theta_0\hat{\sigma}_{ij}, q_i = \rho_0\sqrt{\theta_0}^3\hat{q}_i$$

- 将空间及时间坐标无量纲化，并引入无量纲参数Kn

$$x_i = L\hat{x}_i, t = \frac{L}{\sqrt{\theta_0}}\hat{t}$$

$$Kn = \frac{\mu_0\sqrt{\theta_0}}{\rho_0\theta_0 L}$$

二维R13方程的线性化——方程的降维

- 将方程简化至二维，进行变量名的变换，并且令第三维方向上的各分量为0

$$\hat{x}_1 = \hat{x}, \hat{x}_2 = \hat{y}$$

$$\hat{v}_1 = \hat{v}_x, \hat{v}_2 = \hat{v}_y$$

$$\hat{\sigma}_{11} = \hat{\sigma}_{xx}, \hat{\sigma}_{12} = \hat{\sigma}_{xy}, \hat{\sigma}_{22} = \hat{\sigma}_{yy}$$

$$\hat{q}_1 = \hat{q}_x, \hat{q}_2 = \hat{q}_y$$

$$\hat{v}_3 = \hat{\sigma}_{13} = \hat{\sigma}_{23} = \hat{q}_3 = 0$$

$$\hat{\sigma}_{33} = -\hat{\sigma}_{xx} - \hat{\sigma}_{yy}$$

二维R13方程的线性化——编程计算

- 使用Mathematica进行编程计算，完成方程的化简（包括$\eta=5$和$\eta=10$的情况）

linearization-2D_st.nb * - Wolfram Mathematica 12.0

文件 (F) 编辑 (E) 插入 (I) 格式 (R) 单元 (C) 图形 (G) 计算 (V) 面板 (P) 窗口 (W) 帮助 (H)

```
rulelinear = {ρ[x, y, t] → ρ0 (1 + ϵ * ρ1[x1, y1, t1]), θ[x, y, t] → θ0 (1 + ϵ * θ1[x1, y1, t1]),
   vx[x, y, t] → ϵ √θ0 vx1[x1, y1, t1], vy[x, y, t] → ϵ √θ0 vy1[x1, y1, t1],
   σxx[x, y, t] → ρ0 * θ0 * ϵ * σxx1[x1, y1, t1], σyy[x, y, t] → ρ0 * θ0 * ϵ * σyy1[x1, y1, t1],
   σxy[x, y, t] → ρ0 * θ0 * ϵ * σxy1[x1, y1, t1],
   qx[x, y, t] → ϵ * ρ0 * Power[√θ0, 3] qx1[x1, y1, t1],
   qy[x, y, t] → ϵ * ρ0 * Power[√θ0, 3] qy1[x1, y1, t1]};

rulelinear1 = {ρ^(0,0,1)[x, y, t] -> ρ0 (√θ0/L) ϵ D[ρ1[x1, y1, t1], t1],
   vx^(0,0,1)[x, y, t] → (θ0/L) ϵ D[vx1[x1, y1, t1], t1], vy^(0,0,1)[x, y, t] → (θ0/L) ϵ D[vy1[x1, y1, t1], t1],
```

二维线性化程序的部分截图

二维R13方程的线性化——编程计算

- 使用Mathematica进行编程计算，完成方程的化简（包括$\eta = 5$ 和$\eta = 10$的情况）

linearization-2D_st.nb * - Wolfram Mathematica 12.0

文件 (F) 编辑 (E) 插入 (I) 格式 (R) 单元 (C) 图形 (G) 计算 (V) 面板 (P) 窗口 (W) 帮助 (H)

Out[241]= $\rho1^{(0,0,1)}[x1, y1, t1] + vy1^{(0,1,0)}[x1, y1, t1] + vx1^{(1,0,0)}[x1, y1, t1]$

Out[242]= $vx1^{(0,0,1)}[x1, y1, t1] + \sigma_{xy1}{}^{(0,1,0)}[x1, y1, t1] + \theta1^{(1,0,0)}[x1, y1, t1] + \rho1^{(1,0,0)}[x1, y1, t1] + \sigma_{xx1}{}^{(1,0,0)}[x1, y1, t1]$

Out[243]= $vy1^{(0,0,1)}[x1, y1, t1] + \theta1^{(0,1,0)}[x1, y1, t1] + \rho1^{(0,1,0)}[x1, y1, t1] + \sigma_{yy1}{}^{(0,1,0)}[x1, y1, t1] + \sigma_{xy1}{}^{(1,0,0)}[x1, y1, t1]$

Out[244]= $\theta1^{(0,0,1)}[x1, y1, t1] + \frac{2}{3} qy1^{(0,1,0)}[x1, y1, t1] + \frac{2}{3} vy1^{(0,1,0)}[x1, y1, t1] + \frac{2}{3} qx1^{(1,0,0)}[x1, y1, t1] + \frac{2}{3} vx1^{(1,0,0)}[x1, y1, t1]$

程序的部分输出结果（图中为$\eta = 5$ 时的连续方程、动量方程以及能量方程）

二维R13方程的线性化——线性化结果

- 最终得到二维线性化后的R13方程（仅列出$\eta = 5$ 的情况）

$$\frac{\partial\hat{\rho}}{\partial\hat{t}} + \frac{\partial\hat{v}_x}{\partial\hat{x}} + \frac{\partial\hat{v}_y}{\partial\hat{y}} = 0$$

$$\frac{\partial\hat{v}_x}{\partial\hat{t}} + \frac{\partial\hat{\rho}}{\partial\hat{x}} + \frac{\partial\hat{\theta}}{\partial\hat{x}} + \frac{\partial\hat{\sigma}_{xx}}{\partial\hat{x}} + \frac{\partial\hat{\sigma}_{xy}}{\partial\hat{y}} = 0$$

$$\frac{\partial\hat{v}_y}{\partial\hat{t}} + \frac{\partial\hat{\rho}}{\partial\hat{y}} + \frac{\partial\hat{\theta}}{\partial\hat{y}} + \frac{\partial\hat{\sigma}_{xy}}{\partial\hat{x}} + \frac{\partial\hat{\sigma}_{yy}}{\partial\hat{y}} = 0$$

$$\frac{\partial\hat{\theta}}{\partial\hat{t}} + \frac{2}{3}\frac{\partial\hat{q}_x}{\partial\hat{x}} + \frac{2}{3}\frac{\partial\hat{q}_y}{\partial\hat{y}} + \frac{2}{3}\frac{\partial\hat{v}_x}{\partial\hat{x}} + \frac{2}{3}\frac{\partial\hat{v}_y}{\partial\hat{y}} = 0$$

二维R13方程的线性化——线性化结果

- 最终得到二维线性化后的R13方程（仅列出$\eta = 5$ 的情况）

$$\frac{\partial\hat{\sigma}_{xx}}{\partial\hat{t}} + \frac{8}{15}\frac{\partial\hat{q}_x}{\partial\hat{x}} - \frac{4}{15}\frac{\partial\hat{q}_y}{\partial\hat{y}} + \frac{4}{3}\frac{\partial\hat{v}_x}{\partial\hat{x}} - \frac{2}{3}\frac{\partial\hat{v}_y}{\partial\hat{y}} - \frac{6}{5}Kn\frac{\partial^2\hat{\sigma}_{xx}}{\partial\hat{x}^2} - \frac{2}{3}Kn\frac{\partial^2\hat{\sigma}_{xx}}{\partial\hat{y}^2} - \frac{4}{15}Kn\frac{\partial^2\hat{\sigma}_{xy}}{\partial\hat{x}\partial\hat{y}} + \frac{4}{15}Kn\frac{\partial^2\hat{\sigma}_{yy}}{\partial\hat{y}^2} = -\frac{\hat{\sigma}_{xx}}{Kn}$$

$$\frac{\partial\hat{\sigma}_{xy}}{\partial\hat{t}} + \frac{2}{5}\frac{\partial\hat{q}_x}{\partial\hat{y}} + \frac{2}{5}\frac{\partial\hat{q}_y}{\partial\hat{x}} + \frac{\partial\hat{v}_x}{\partial\hat{y}} + \frac{\partial\hat{v}_y}{\partial\hat{x}} - \frac{16}{15}Kn\frac{\partial^2\hat{\sigma}_{xy}}{\partial\hat{x}^2} - \frac{16}{15}Kn\frac{\partial^2\hat{\sigma}_{xy}}{\partial\hat{y}^2} - \frac{2}{5}Kn\frac{\partial^2\hat{\sigma}_{xx}}{\partial\hat{x}\partial\hat{y}} - \frac{2}{5}Kn\frac{\partial^2\hat{\sigma}_{yy}}{\partial\hat{x}\partial\hat{y}} = -\frac{\hat{\sigma}_{xy}}{Kn}$$

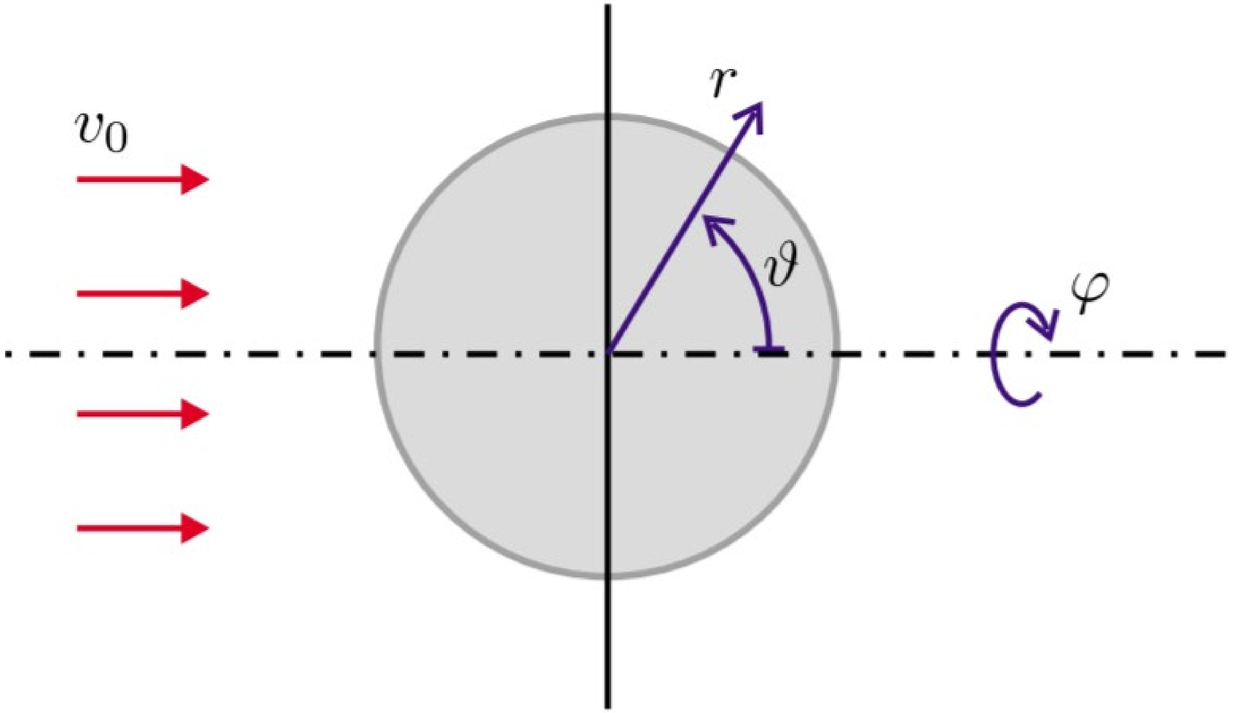
$$\frac{\partial\hat{\sigma}_{yy}}{\partial\hat{t}} + \frac{8}{15}\frac{\partial\hat{q}_y}{\partial\hat{y}} - \frac{4}{15}\frac{\partial\hat{q}_x}{\partial\hat{x}} + \frac{4}{3}\frac{\partial\hat{v}_y}{\partial\hat{y}} - \frac{2}{3}\frac{\partial\hat{v}_x}{\partial\hat{x}} - \frac{6}{5}Kn\frac{\partial^2\hat{\sigma}_{yy}}{\partial\hat{y}^2} - \frac{2}{3}Kn\frac{\partial^2\hat{\sigma}_{yy}}{\partial\hat{x}^2} - \frac{4}{15}Kn\frac{\partial^2\hat{\sigma}_{xy}}{\partial\hat{x}\partial\hat{y}} + \frac{4}{15}Kn\frac{\partial^2\hat{\sigma}_{xx}}{\partial\hat{x}^2} = -\frac{\hat{\sigma}_{yy}}{Kn}$$

$$\frac{\partial\hat{q}_x}{\partial\hat{t}} + \frac{\partial\hat{\sigma}_{xx}}{\partial\hat{x}} + \frac{\partial\hat{\sigma}_{xy}}{\partial\hat{y}} + \frac{5}{2}\frac{\partial\hat{\theta}}{\partial\hat{x}} - 3.6Kn\frac{\partial^2\hat{q}_x}{\partial\hat{x}^2} - 1.2Kn\frac{\partial^2\hat{q}_x}{\partial\hat{y}^2} - 2.4Kn\frac{\partial^2\hat{q}_y}{\partial\hat{x}\partial\hat{y}} = -\frac{2}{3}\frac{\hat{q}_x}{Kn}$$

$$\frac{\partial\hat{q}_y}{\partial\hat{t}} + \frac{\partial\hat{\sigma}_{xy}}{\partial\hat{x}} + \frac{\partial\hat{\sigma}_{yy}}{\partial\hat{y}} + \frac{5}{2}\frac{\partial\hat{\theta}}{\partial\hat{y}} - 3.6Kn\frac{\partial^2\hat{q}_y}{\partial\hat{y}^2} - 1.2Kn\frac{\partial^2\hat{q}_y}{\partial\hat{x}^2} - 2.4Kn\frac{\partial^2\hat{q}_x}{\partial\hat{x}\partial\hat{y}} = -\frac{2}{3}\frac{\hat{q}_y}{Kn}$$

2. 工作内容——圆球绕流求解

- 针对圆球绕流问题，尝试求R13方程的解析解
- 圆球绕流：三维轴对称问题，流场关于角度φ对称，即流动参量与φ无关



圆球绕流求解——坐标变换及化简

- 先将方程从直角坐标系转到球坐标系，再对方程进行φ方向的化简

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$\frac{\partial r}{\partial x} = \sin\vartheta\cos\phi,\ \frac{\partial \vartheta}{\partial x} = \frac{\cos\vartheta\cos\phi}{r},\ \frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r\sin\vartheta}$$

$$\frac{\partial r}{\partial y} = \sin\vartheta\sin\phi,\ \frac{\partial \vartheta}{\partial y} = \frac{\cos\vartheta\sin\phi}{r},\ \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r\sin\vartheta}$$

$$\frac{\partial r}{\partial z} = \cos\vartheta,\ \frac{\partial \vartheta}{\partial z} = -\frac{\sin\vartheta}{r},\ \frac{\partial \phi}{\partial z} = 0$$

$$\mathbf{F} = F_1\,\boldsymbol{i} + F_2\,\boldsymbol{j} + F_3\,\boldsymbol{k} = F_r\,\mathbf{e}_r + F_\vartheta\,\mathbf{e}_\vartheta + F_\phi\,\mathbf{e}_\phi$$

$$F_1 = \sin\vartheta\cos\phi F_r + \cos\vartheta\cos\phi F_\vartheta - \sin\phi F_\phi$$

$$F_2 = \sin\vartheta\sin\phi F_r + \cos\vartheta\sin\phi F_\vartheta + \cos\phi F_\phi$$

$$F_3 = \cos\vartheta F_r - \sin\vartheta F_\vartheta$$

化简规则：

$$\partial/\partial\phi = 0,\ \partial^m / \partial^{m1} r\,\partial^{m2}\vartheta\,\partial^{m3}\phi = 0,\ \text{if}\ (m3 > 0)$$

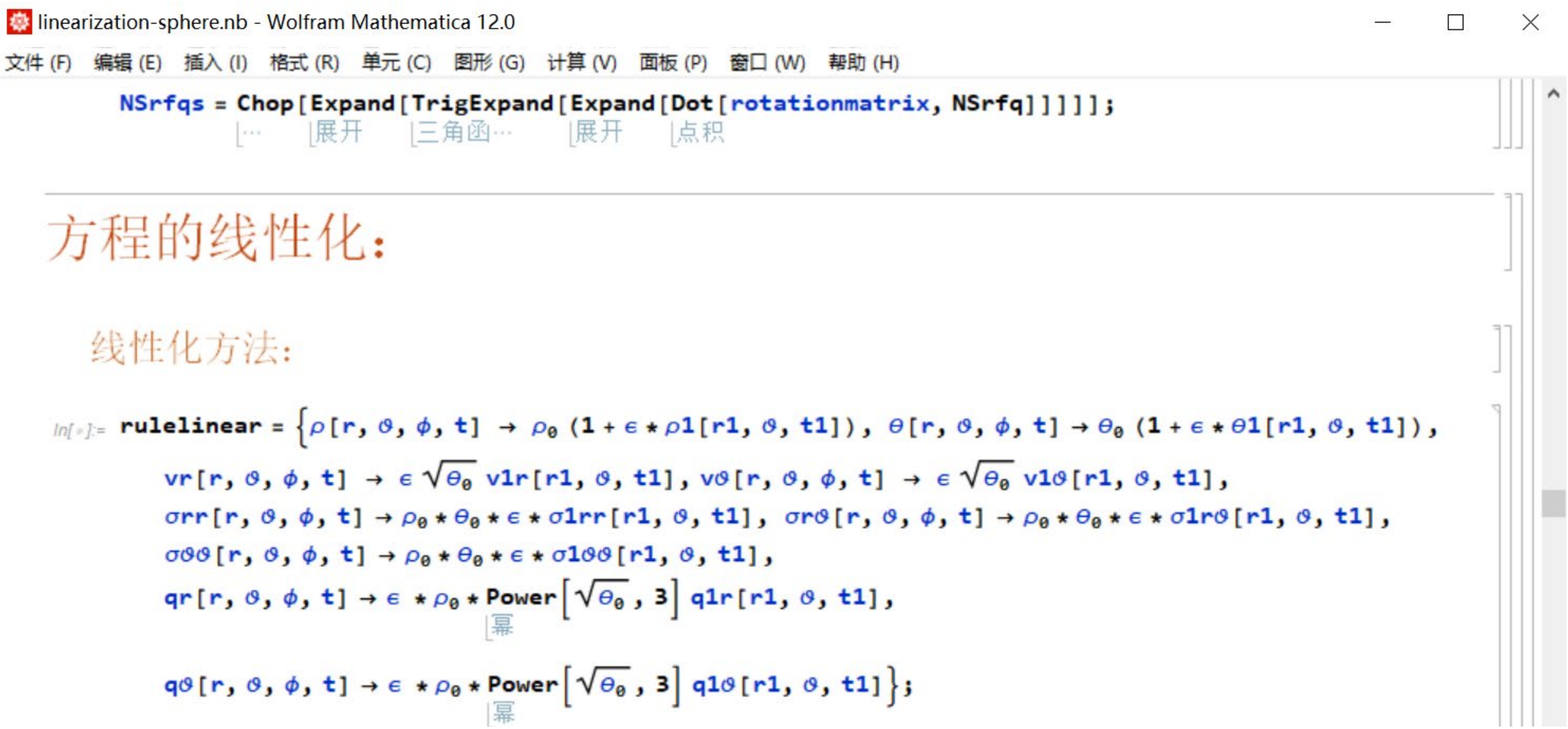
$$v_\phi = 0,\ q_\phi = 0;\ \sigma_{ij} = 0,\ \text{if}\ (i \neq j,\ i == \phi\ ||\ j == \phi)$$

$$\sigma_{rr} + \sigma_{\vartheta\vartheta} + \sigma_{\phi\phi} = 0$$

程序中的部分注释，展示了坐标变换和化简的方法

圆球绕流求解——方程的线性化

- 依据与之前类似的准则，同样对方程进行无量纲线性化



程序线性化部分的截图

圆球绕流求解——转换待求解参量

- 对照Stokes流动的表达式，给定各流动参量的表达式所具有的形式
- a, b, c, d, α, β, γ, κ等都是只以r/R这一无量纲参数为自变量的函数

$$\mathbf{v}(r,\vartheta)=v_0\begin{pmatrix}\left(1+a\left(\frac{r}{R}\right)\right)\cos\vartheta\\ -\left(1+b\left(\frac{r}{R}\right)\right)\sin\vartheta\\ 0\end{pmatrix}$$

$$\mathbf{q}(r,\vartheta)=p_0\sqrt{\theta_0}\begin{pmatrix}\alpha\left(\frac{r}{R}\right)\cos\vartheta\\ -\beta\left(\frac{r}{R}\right)\sin\vartheta\\ 0\end{pmatrix},$$

$$\theta(r,\vartheta)=\theta_0 c\left(\frac{r}{R}\right)\cos\vartheta,\quad p(r,\vartheta)=p_0 d\left(\frac{r}{R}\right)\cos\vartheta$$

$$\boldsymbol{\sigma}(r,\vartheta)=p_0\begin{pmatrix}\gamma\left(\frac{r}{R}\right)\cos\vartheta & \kappa\left(\frac{r}{R}\right)\sin\vartheta & 0\\ \kappa\left(\frac{r}{R}\right)\sin\vartheta & \delta\left(\frac{r}{R}\right)\cos\vartheta & 0\\ 0 & 0 & \sigma_{\varphi\varphi}\end{pmatrix}$$

圆球绕流求解——转换待求解参量

- 将给定的形式代入，将原方程组变为a, b, c, d, α, β, γ, κ等函数为未知量的方程组（$M_0 = v_0/\sqrt{\theta_0}$）

Out[•]= $\frac{2\,a[r1]}{r1} - \frac{2\,b[r1]}{r1} + a'[r1]$

Out[•]= $\frac{3\,\gamma[r1]}{r1} + \frac{2\,\kappa[r1]}{r1} + d'[r1] + \gamma'[r1]$

Out[•]= $-\frac{d[r1]}{r1} + \frac{\gamma[r1]}{2\,r1} + \frac{3\,\kappa[r1]}{r1} + \kappa'[r1]$

Out[•]= $\frac{2\,\alpha[r1]}{r1} - \frac{2\,\beta[r1]}{r1} + \alpha'[r1]$

Out[•]= $\frac{2\,M0\,a[r1]}{3\,r1} - \frac{2\,M0\,b[r1]}{3\,r1} + \frac{4\,\alpha[r1]}{15\,r1} - \frac{4\,\beta[r1]}{15\,r1} - \frac{\gamma[r1]}{2\,Kn} - \frac{22\,Kn\,\gamma[r1]}{5\,r1^2} - \frac{68\,Kn\,\kappa[r1]}{15\,r1^2} - \frac{2}{3}\,M0\,a'[r1] - \frac{4\,\alpha'[r1]}{15} + \frac{6\,Kn\,\gamma'[r1]}{5\,r1} + \frac{4\,Kn\,\kappa'[r1]}{15\,r1} + \frac{3}{5}\,Kn\,\gamma''[r1]$

Out[•]= $\frac{M0\,a[r1]}{2\,r1} - \frac{M0\,b[r1]}{2\,r1} + \frac{\alpha[r1]}{5\,r1} - \frac{\beta[r1]}{5\,r1} - \frac{9\,Kn\,\gamma[r1]}{5\,r1^2} - \frac{\kappa[r1]}{2\,Kn} - \frac{18\,Kn\,\kappa[r1]}{5\,r1^2} + \frac{1}{2}\,M0\,b'[r1] + \frac{\beta'[r1]}{5} - \frac{Kn\,\gamma'[r1]}{10\,r1} + \frac{16\,Kn\,\kappa'[r1]}{15\,r1} + \frac{8}{15}\,Kn\,\kappa''[r1]$

Out[•]= $-\frac{M0\,a[r1]}{3\,r1} + \frac{M0\,b[r1]}{3\,r1} - \frac{2\,\alpha[r1]}{15\,r1} + \frac{2\,\beta[r1]}{15\,r1} + \frac{\gamma[r1]}{4\,Kn} + \frac{11\,Kn\,\gamma[r1]}{5\,r1^2} + \frac{34\,Kn\,\kappa[r1]}{15\,r1^2} + \frac{1}{3}\,M0\,a'[r1] + \frac{2\,\alpha'[r1]}{15} - \frac{3\,Kn\,\gamma'[r1]}{5\,r1} - \frac{2\,Kn\,\kappa'[r1]}{15\,r1} - \frac{3}{10}\,Kn\,\gamma''[r1]$

Out[•]= $\frac{2\,\alpha[r1]}{3\,Kn} + \frac{48\,Kn\,\alpha[r1]}{5\,r1^2} - \frac{48\,Kn\,\beta[r1]}{5\,r1^2} + \frac{3\,\gamma[r1]}{r1} + \frac{2\,\kappa[r1]}{r1} + \frac{5\,c'[r1]}{2} - \frac{36\,Kn\,\alpha'[r1]}{5\,r1} + \frac{24\,Kn\,\beta'[r1]}{5\,r1} + \gamma'[r1] - \frac{18}{5}\,Kn\,\alpha''[r1]$

Out[•]= $\frac{5\,c[r1]}{4\,r1} - \frac{18\,Kn\,\alpha[r1]}{5\,r1^2} + \frac{\beta[r1]}{3\,Kn} + \frac{18\,Kn\,\beta[r1]}{5\,r1^2} - \frac{\gamma[r1]}{4\,r1} - \frac{3\,\kappa[r1]}{2\,r1} - \frac{6\,Kn\,\alpha'[r1]}{5\,r1} - \frac{6\,Kn\,\beta'[r1]}{5\,r1} - \frac{\kappa'[r1]}{2} - \frac{3}{5}\,Kn\,\beta''[r1]$

输出的转换未知量后方程（$\eta = 5$ 的情况）

圆球绕流求解——假设解的形式

- 根据克努森层理论以及方程中二阶导数项的个数， $\eta=5$ 时a, b, c, d, α, β, γ, κ具有的通用形式为：

$$\psi(x)=\psi_0(x)+\psi_1(x)\exp[-\lambda_1(x-1)]+\psi_2(x)\exp[-\lambda_2(x-1)]$$

- 其中$\psi_i(x)$ 可以写为x的负指数次幂的组合（求解时n取8）：

$$\psi_i(x)=\sum_{j=1}^{n}\frac{C_{ij}}{x^j}$$

- 根据方程中的系数，两个常数的取值为：

$$\lambda_1=\frac{1}{Kn}\sqrt{\frac{5}{9}},\lambda_2=\frac{1}{Kn}\sqrt{\frac{5}{6}}$$

圆球绕流求解——代入求通解

- 将假设的解的形式代入方程，令各式中线性无关的各项前系数均为0，即将微分方程转化为代数方程，可求出各系数，得到方程组的通解

Analytical(2).nb - Wolfram Mathematica 12.0

文件 (F) 编辑 (E) 插入 (I) 格式 (R) 单元 (C) 图形 (G) 计算 (V) 面板 (P) 窗口 (W) 帮助 (H)

Solve: Equations may not give solutions for all "solve" variables.

Out[•]=

$$\Big\{\Big\{a0[1]\to\frac{M0\,\gamma0[2]}{2\,Kn},\ a0[2]\to0,\ a0[3]\to-\frac{2}{5}M0\,\alpha0[3]+\frac{16}{15}Kn\,M0\,\gamma0[2]+\frac{M0\,\gamma0[4]}{6\,Kn},\ a0[4]\to0,\ a0[5]\to0,\ a0[6]\to0,$$

$$a0[7]\to0,\ a0[8]\to0,\ a1[1]\to0,\ a1[2]\to-\frac{2}{5}M0\,\alpha1[2],\ a1[3]\to-\frac{6\,Kn\,M0\,\alpha1[2]}{5\sqrt{5}},\ a1[4]\to0,\ a1[5]\to0,\ a1[6]\to0,$$

$$a1[7]\to0,\ a1[8]\to0,\ b0[1]\to\frac{M0\,\gamma0[2]}{4\,Kn},\ b0[2]\to0,\ b0[3]\to\frac{1}{5}M0\,\alpha0[3]-\frac{8}{15}Kn\,M0\,\gamma0[2]-\frac{M0\,\gamma0[4]}{12\,Kn},\ b0[4]\to0,$$

$$b0[5]\to0,\ b0[6]\to0,\ b0[7]\to0,\ b0[8]\to0,\ b1[1]\to\frac{M0\,\alpha1[2]}{3\sqrt{5}\,Kn},\ b1[2]\to\frac{1}{5}M0\,\alpha1[2],\ b1[3]\to\frac{3\,Kn\,M0\,\alpha1[2]}{5\sqrt{5}},\ b1[4]\to0,$$

$$b1[5]\to0,\ b1[6]\to0,\ b1[7]\to0,\ b1[8]\to0,\ \alpha0[1]\to0,\ \alpha0[2]\to0,\ \alpha0[4]\to0,\ \alpha0[5]\to0,\ \alpha0[6]\to0,\ \alpha0[7]\to0,$$

$$\alpha0[8]\to0,\ \alpha1[1]\to0,\ \alpha1[3]\to\frac{3\,Kn\,\alpha1[2]}{\sqrt{5}},\ \alpha1[4]\to0,\ \alpha1[5]\to0,\ \alpha1[6]\to0,\ \alpha1[7]\to0,\ \alpha1[8]\to0,\ \beta0[1]\to0,\ \beta0[2]\to0,$$

$$\beta0[3]\to-\frac{\alpha0[3]}{2},\ \beta0[4]\to0,\ \beta0[5]\to0,\ \beta0[6]\to0,\ \beta0[7]\to0,\ \beta0[8]\to0,\ \beta1[1]\to-\frac{\sqrt{5}\,\alpha1[2]}{6\,Kn},\ \beta1[2]\to-\frac{\alpha1[2]}{2},$$

$$\beta1[3]\to-\frac{3\,Kn\,\alpha1[2]}{2\sqrt{5}},\ \beta1[4]\to0,\ \beta1[5]\to0,\ \beta1[6]\to0,\ \beta1[7]\to0,\ \beta1[8]\to0,\ c0[1]\to0,\ c0[2]\to\frac{2\,\alpha0[3]}{15\,Kn}+\frac{\gamma0[2]}{5},$$

解出的一部分系数

圆球绕流求解——边界条件

- 调研文献，圆球绕流问题的边条件为

$$
\begin{aligned}
&-\kappa(1)=\tilde{\chi}\left(-M_0(1+b)-\tfrac{1}{5}\beta+\tfrac{8}{15}\mathrm{Kn}\left(\tfrac{3}{4}\gamma-\kappa'+2\kappa\right)\right)_{x=1},\\
&-\alpha(1)=\tilde{\chi}\left(2c+\tfrac{1}{2}\gamma-\tfrac{6}{7}\mathrm{Kn}\alpha'\right)_{x=1},\\
&\tfrac{6}{15}\mathrm{Kn}(3\gamma'-6\gamma-4\kappa)_{x=1}=\tilde{\chi}\left(-\tfrac{2}{5}c+\tfrac{7}{5}\gamma-\tfrac{12}{35}\mathrm{Kn}\alpha'\right)_{x=1},\\
&\tfrac{12}{5}\mathrm{Kn}(\beta-\alpha-\beta')_{x=1}\\
&\qquad=\tilde{\chi}\left(M_0(1+b)-\tfrac{11}{5}\beta+\tfrac{8}{15}\mathrm{Kn}\left(\tfrac{3}{4}\gamma-\kappa'+2\kappa\right)\right)_{x=1}.\\
&1+a(1)=0,
\end{aligned}
$$

- 将边条件代入通解，即求得该问题完整的解

3.成果展示与总结

方程组的通解

$$a(x) = C_1 \frac{1}{2x} + C_2 \frac{1}{3x^3} - K_1 \left(\frac{6Kn^3}{5x^3} + \frac{2}{\sqrt{5}} \frac{Kn^2}{x^2} \right) \exp\left[-\frac{1}{Kn} \sqrt{\frac{5}{9}} (x-1) \right]$$

$$b(x) = C_1 \frac{1}{4x} - C_2 \frac{1}{6x^3} + K_1 \left(\frac{3Kn^3}{5x^3} + \frac{1}{\sqrt{5}} \frac{Kn^2}{x^2} + \frac{Kn}{6x} \right) \exp\left[-\frac{1}{Kn} \sqrt{\frac{5}{9}} (x-1) \right]$$

$$\frac{\alpha(x)}{M_0} = C_3 \frac{Kn}{6x^3} + K_1 \left(\frac{3Kn^3}{x^3} + \sqrt{5} \frac{Kn^2}{x^2} \right) \exp\left[-\frac{1}{Kn} \sqrt{\frac{5}{9}} (x-1) \right]$$

$$\frac{\beta(x)}{M_0} = -C_3 \frac{Kn}{12x^3} - K_1 \left(\frac{3Kn^3}{2x^3} + \sqrt{5} \frac{Kn^2}{2x^2} + \frac{5Kn}{6x} \right) \exp\left[-\frac{1}{Kn} \sqrt{\frac{5}{9}} (x-1) \right]$$

方程组的通解

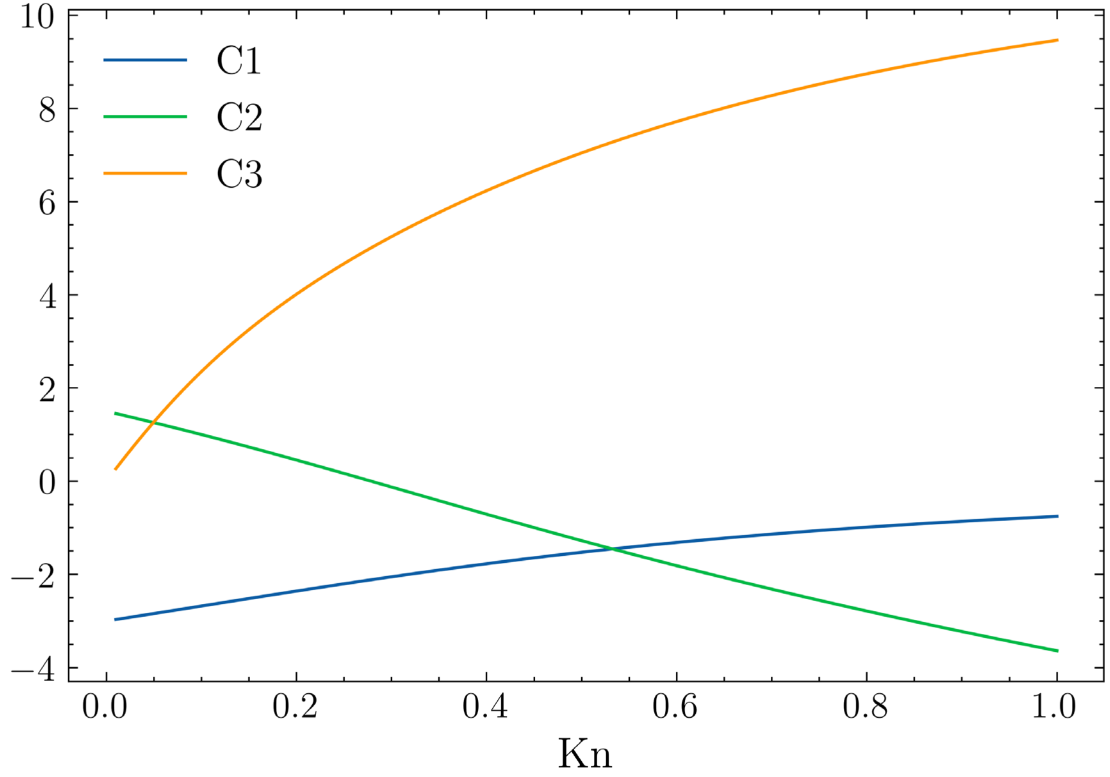
$$\frac{c(x)}{M_0} = (9C_1Kn + C_3)\frac{1}{45x^2} + K_2\left(\frac{2\sqrt{30}}{5}\frac{Kn^2}{x^2} + \frac{2Kn}{x}\right)\exp\left[-\frac{1}{Kn}\sqrt{\frac{5}{6}}(x-1)\right]$$

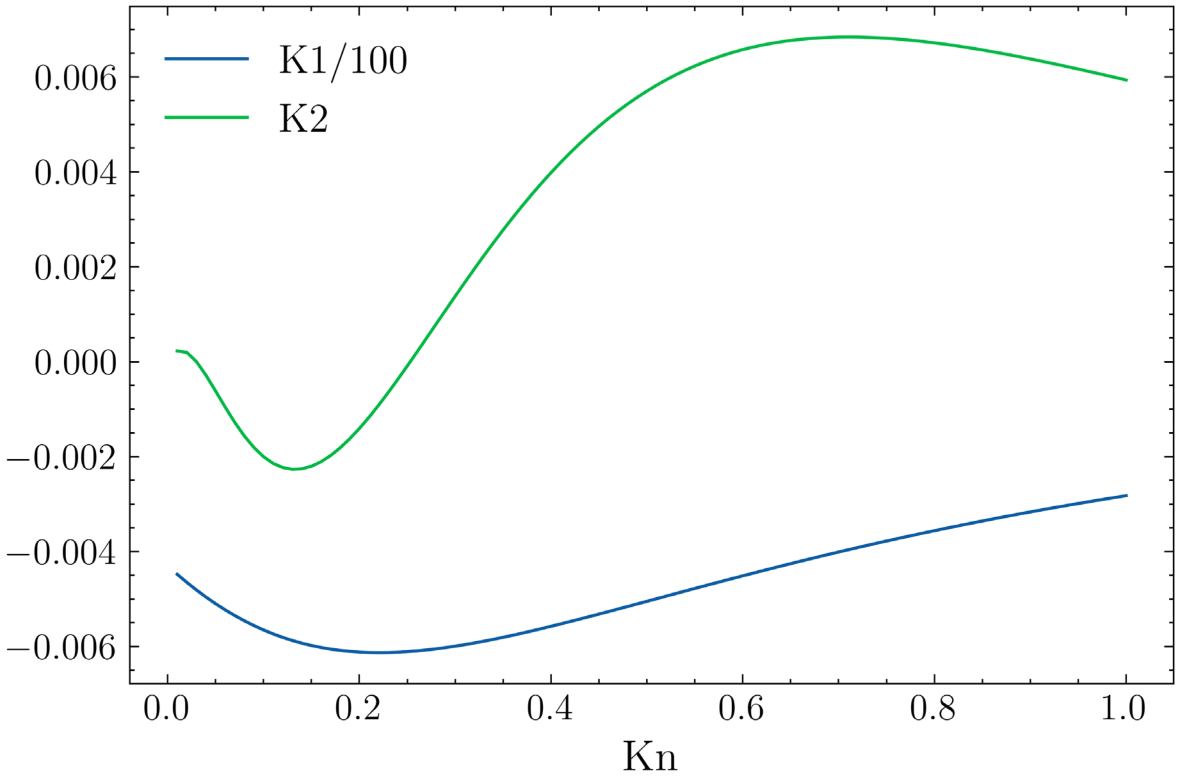
$$\frac{d(x)}{M_0} = C_1\frac{1}{2x^2} + K_2\left(\sqrt{30}\frac{Kn^2}{x^2} + \frac{5Kn}{x}\right)\exp\left[-\frac{1}{Kn}\sqrt{\frac{5}{6}}(x-1)\right]$$

$$\frac{\gamma(x)}{M_0} = (-16C_1Kn^2 + 5C_2 + C_3Kn)\frac{2Kn}{5x^4} + C_1\frac{Kn}{x^2} - K_2\left(\frac{54\sqrt{30}}{5}\frac{Kn^4}{x^4} + 54\frac{Kn^3}{x^3} + 4\sqrt{30}\frac{Kn^2}{x^2} + \frac{5Kn}{x}\right)\exp\left[-\frac{1}{Kn}\sqrt{\frac{5}{6}}(x-1)\right]$$

$$\frac{\kappa(x)}{M_0} = (-16C_1Kn^2 + 5C_2 + C_3Kn)\frac{Kn}{5x^4} - K_2\left(\frac{27\sqrt{30}}{5}\frac{Kn^4}{x^4} + 27\frac{Kn^3}{x^3} + \frac{3\sqrt{30}}{2}\frac{Kn^2}{x^2}\right)\exp\left[-\frac{1}{Kn}\sqrt{\frac{5}{6}}(x-1)\right]$$

积分常数求解结果　　　左：体积积分系数 右：克努森层积分系数

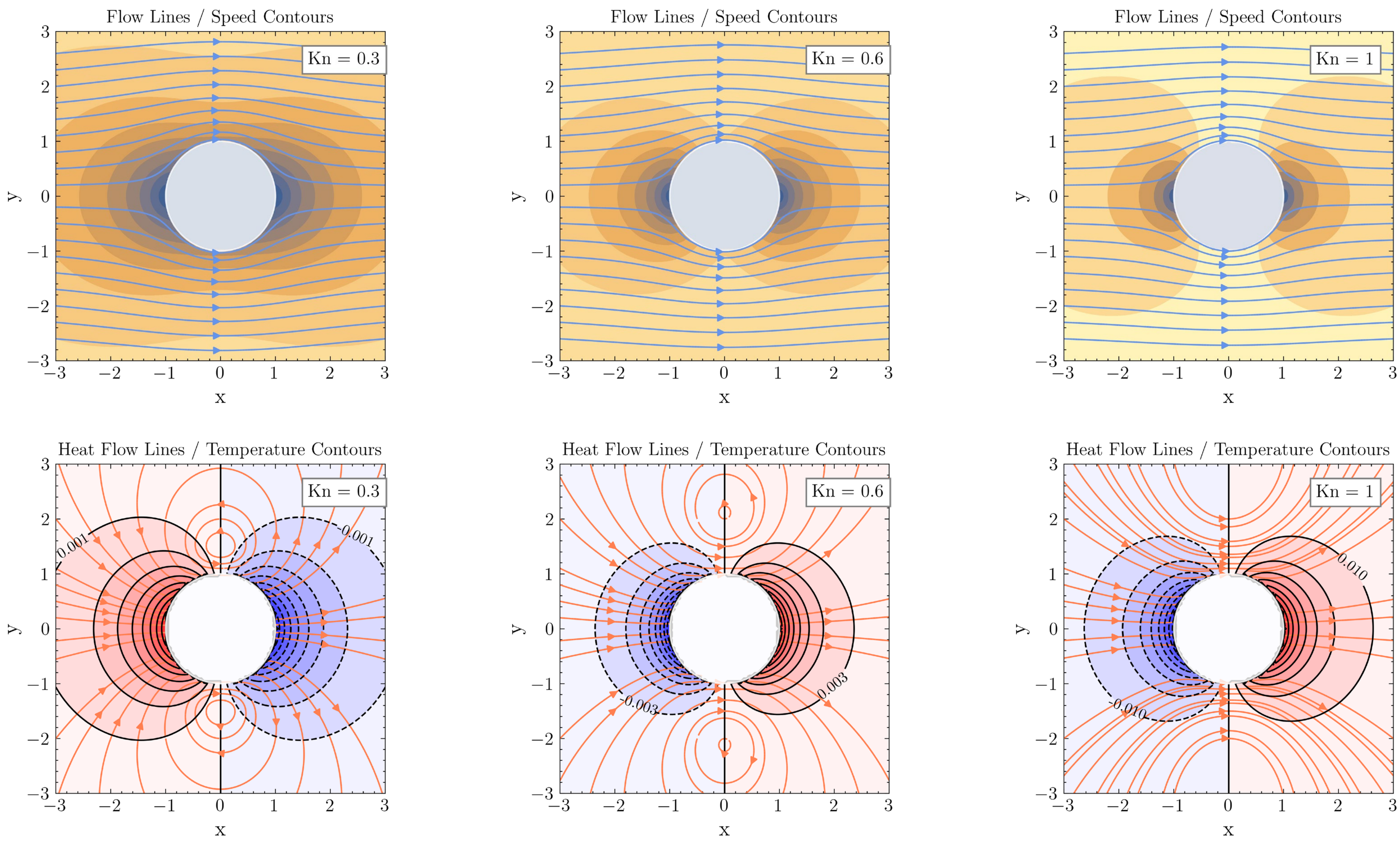




1. 当$Kn \to 0$时，$K1 = -0.485, K2 = 0$
2. $K1$所对应的克努森层的解的系数均包含Kn数的幂

➢ $Kn \to 0$时，克努森层引入的指数解消失

不同克努森数下R13方程流场及温度场



stokes方程+混合边界条件

$$a(x)=C_1\frac{1}{2x}+C_2\frac{1}{3x^3},\quad b(x)=C_1\frac{1}{4x}-C_2\frac{1}{6x^3}$$

$$\frac{\alpha(x)}{M_0}=C_3\frac{\mathrm{Kn}}{6x^3},\quad \frac{\beta(x)}{M_0}=-C_3\frac{\mathrm{Kn}}{12x^3}$$

$$\frac{c(x)}{M_0}=C_3\frac{1}{45x^2},\quad \frac{d(x)}{M_0}=C_1\frac{\mathrm{Kn}}{2x^2}$$

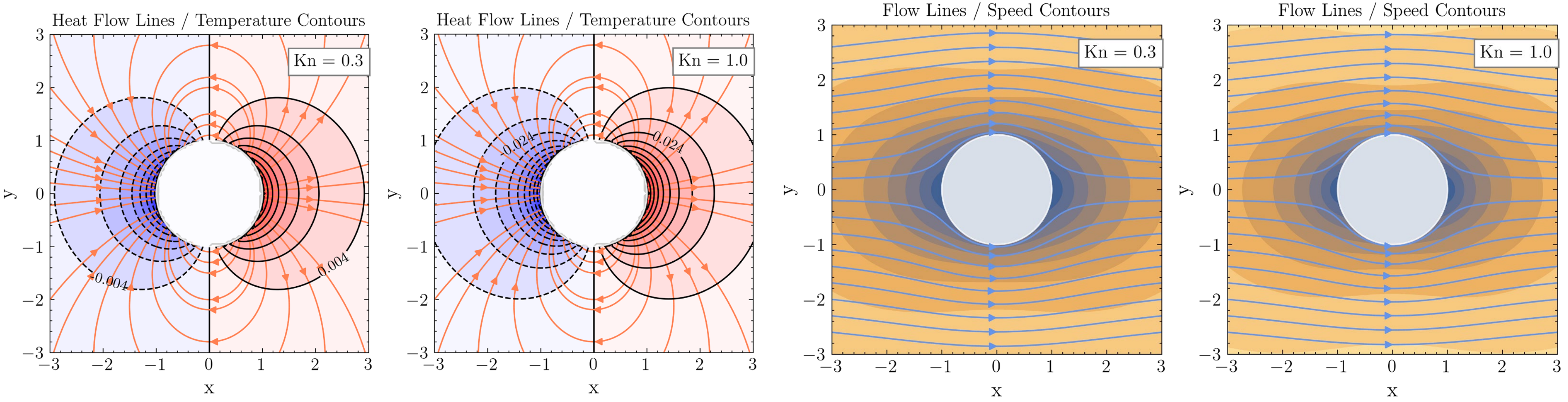
$$\frac{\gamma(x)}{M_0}=C_2\frac{2\mathrm{Kn}}{x^4}+C_1\frac{\mathrm{Kn}}{x^2},\quad \frac{\kappa(x)}{M_0}=C_2\frac{\mathrm{Kn}}{x^4}$$

$$\begin{aligned} v_\vartheta^{(\mathrm{slip})} &= -\frac{\sqrt{\theta_0}}{\tilde{\chi}p_0}\sigma_{r\partial}^{(\mathrm{NSF})}-\frac{1}{5p_0}q_\vartheta^{(\mathrm{NSF})} \\ \theta^{(jump)} &= -\frac{\sqrt{\theta_0}}{2\tilde{\chi}p_0}q_r^{(\mathrm{NSF})}-\frac{\theta_0}{4p_0}\sigma_{rr}^{(\mathrm{NSF})} \end{aligned}$$

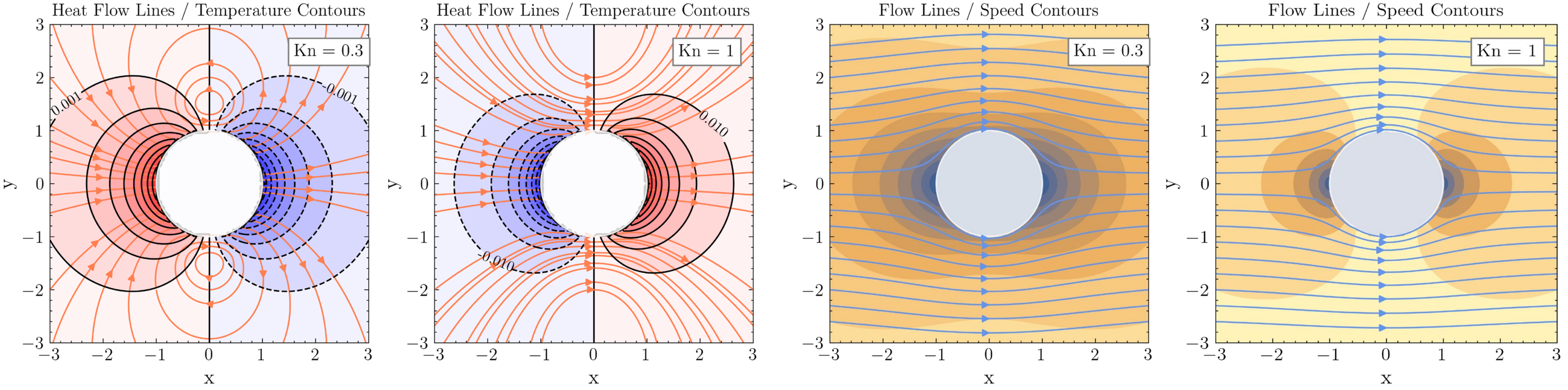
$$\begin{aligned} -\kappa(1) &= \tilde{\chi}\left[-M_0(1+b)-\tfrac{1}{5}\beta\right]_{x=1} \\ -\alpha(1) &= \tilde{\chi}\left(2c+\tfrac{1}{2}\gamma\right)_{x=1} \end{aligned}$$

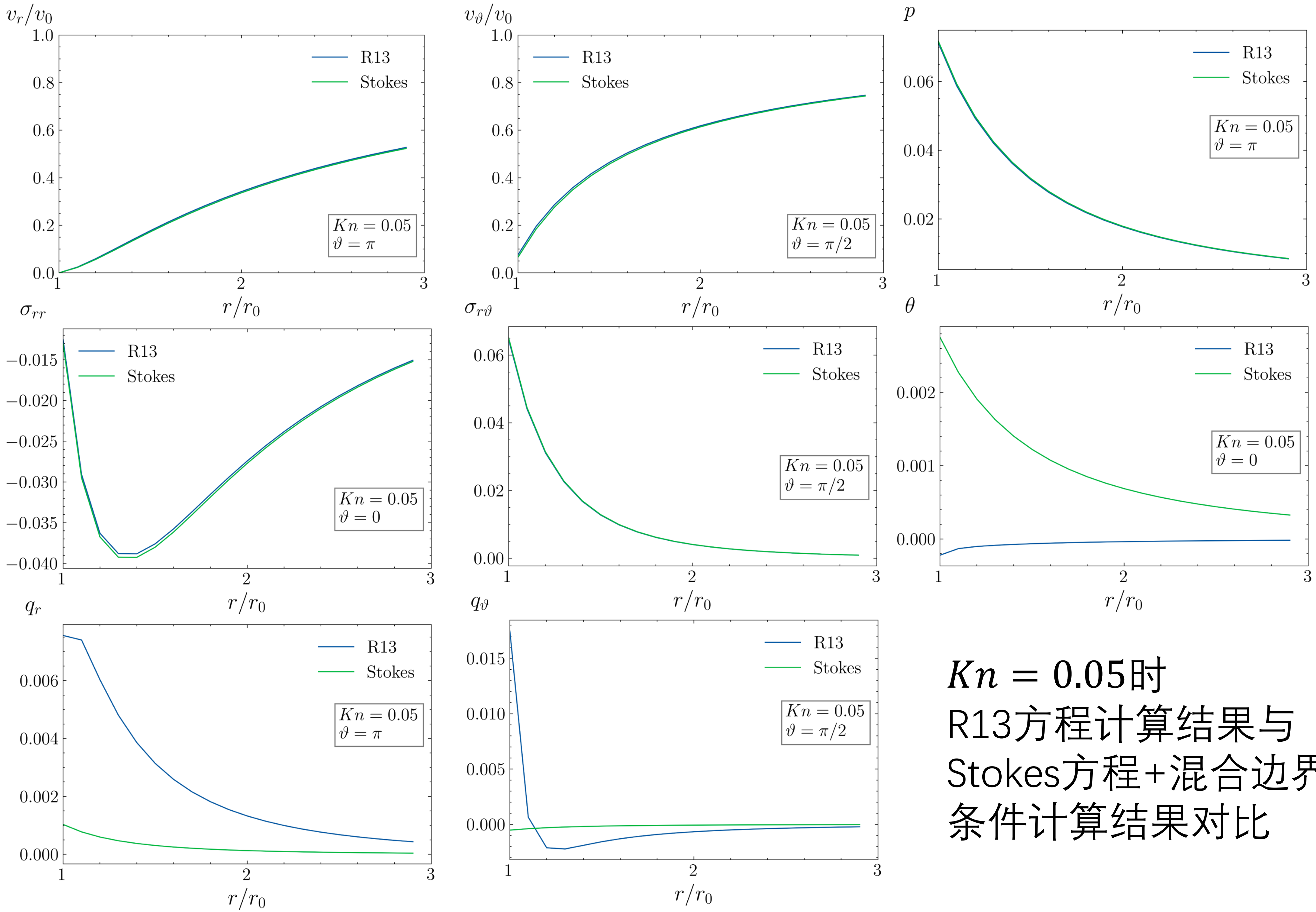
流场对比

Stokes方程+
混合边界条件:

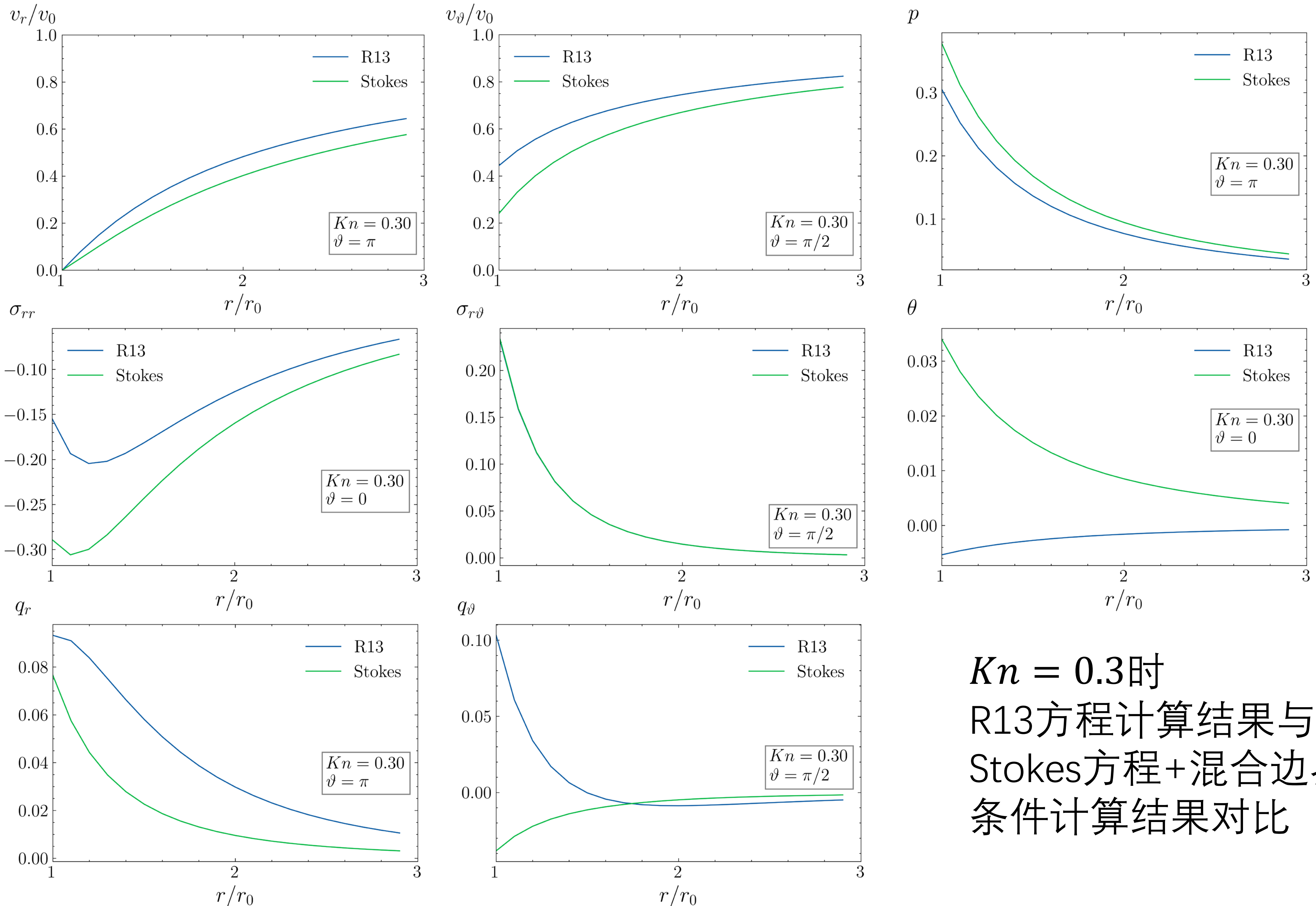


R13方程:

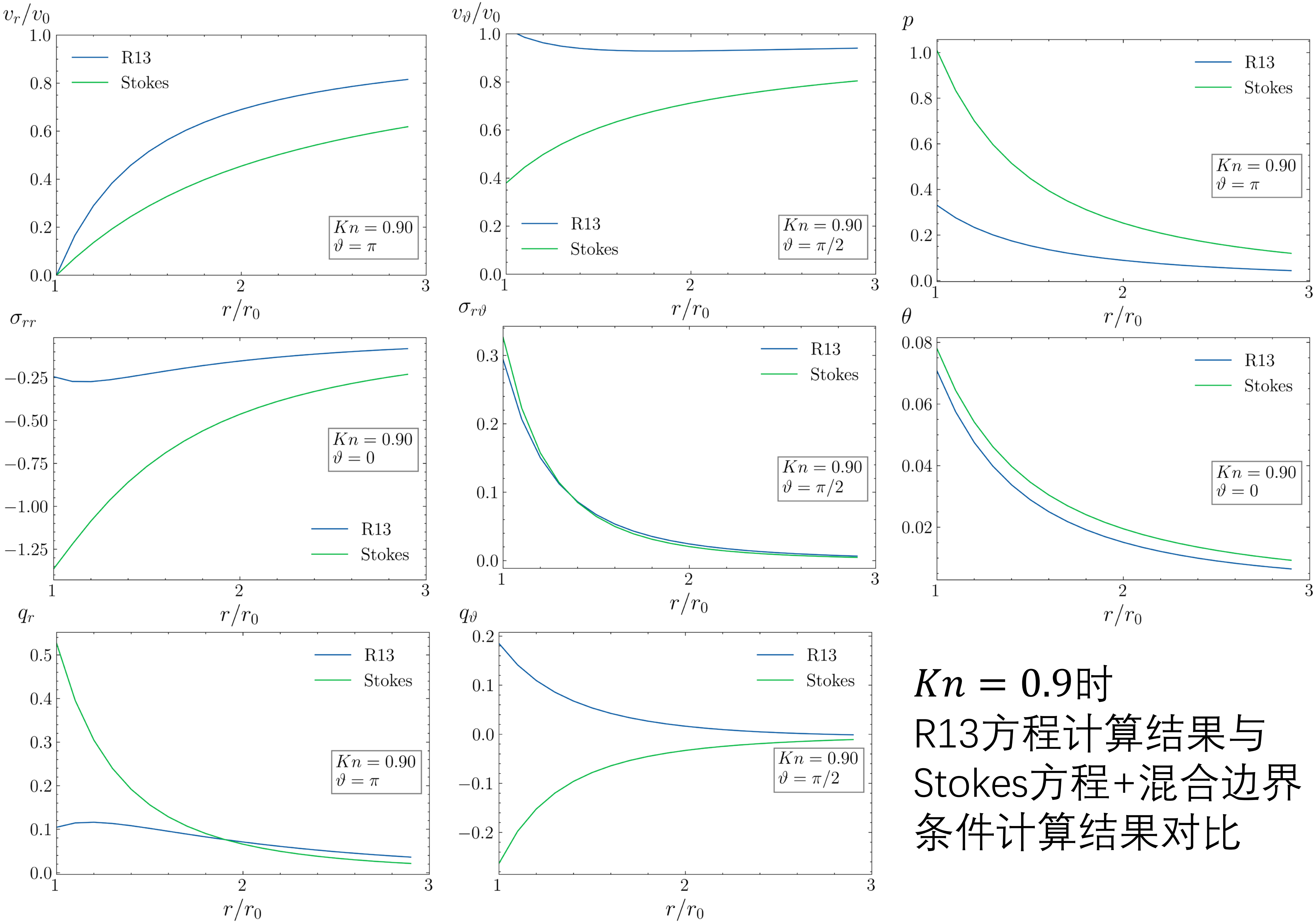




Kn = 0.05时
R13方程计算结果与
Stokes方程+混合边界
条件计算结果对比

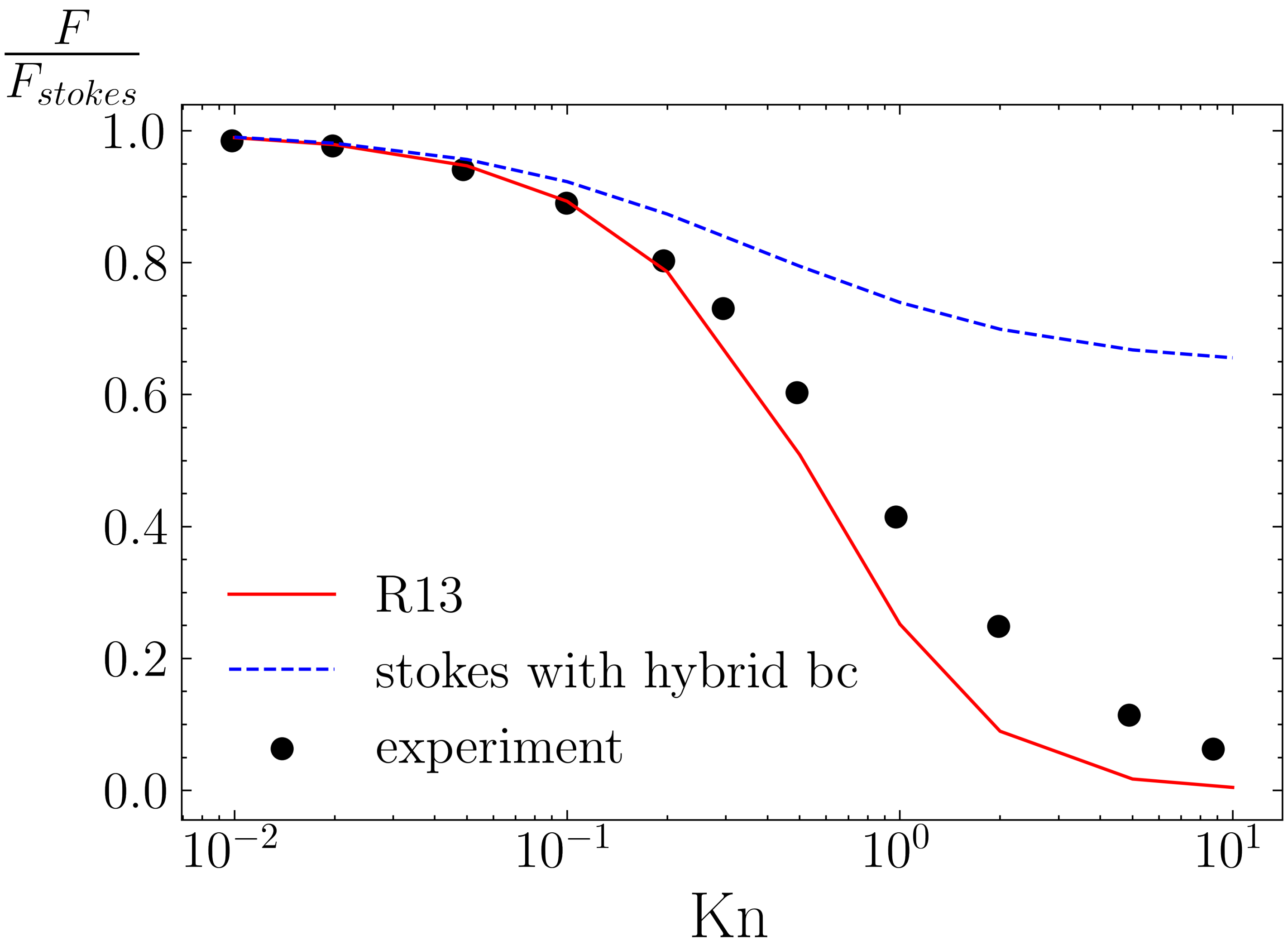


$Kn = 0.3$时
R13方程计算结果与
Stokes方程+混合边界
条件计算结果对比



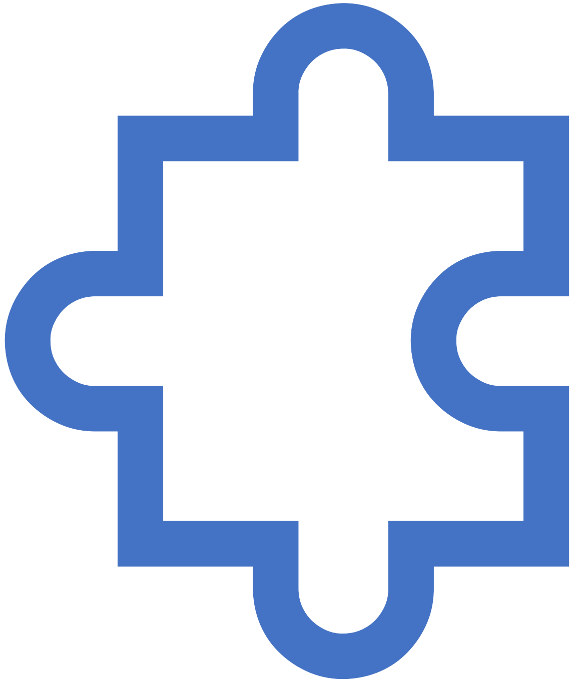
$Kn = 0.9$时
R13方程计算结果与Stokes方程+混合边界条件计算结果对比

不同克努森数下的扰流阻力



预期工作

- 针对$\eta \neq 5$时(非 Maxwell 分子)，给出合理的边界条件
- 选取有效的数值算法进行数值求解



谢谢！